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WATER RESOURCES SYSTEMS ANALYSIS

Part IV. Review of Programming Techniques

Metz Reference Room
Civil Engineering Department
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PREFACE

Water Resources Systems Analysis is a major program of teaching and research in the hydraulic engineering area of the Department of Civil Engineering in the University of Illinois at Urbana, Illinois. For this program, several research projects are being conducted. Three of these projects are supported by the U.S. Department of the Interior, Office of Water Resources Research: OWRR Project No. B-030-ILL on 'Advanced Methodologies for Water Resources Planning,' OWRR Project No. A-029-ILL on 'Stochastic Analysis of Hydrologic Systems,' and OWRR Project No. B-038-ILL on 'Stochastic Analysis of Hydrologic Systems - Phase II.'

Hydrologic data serve as an input to the water resources system. The natural hydrologic processes, which produce the hydrologic data, are truly 'stochastic' in the sense that natural hydrologic phenomena change with time in accordance with the law of probability as well as with the sequential relationship between their occurrences. In OWRR Project Nos. A-029-ILL and B-038-ILL, the objective of the research is to develop a practical procedure by which the stochastic behavior of the hydrologic characteristics of a hydrologic system is to be adequately simulated mathematically. The initial step of this research involved a comprehensive review of the application of the theory of stochastic processes in hydrology. As a result, the following two reports were produced:

'Water Resources Systems Analysis: Part I.
Annotated Bibliography on Stochastic Processes'

'Water Resources Systems Analysis: Part III.
Review of Stochastic Processes'

In OWRR Project No. B-030-ILL, advanced methodologies for water resources planning deal essentially with the mathematical formulation of hydroeconomic systems and the various programming techniques for the optimization of

the systems. The initial literature research for this project produced the following two reports:

'Water Resources Systems Analysis: Part II.
Annotated Bibliography on Programming Techniques'

'Water Resources Systems Analysis: Part IV.
Review of Programming Techniques'

In preparing the reports on annotated bibliographies, only references which have direct bearing on research objectives were selected. The bibliographies are by no means complete, but they cover most items of basic significance to the subject matter and thus should provide a valuable source of information to anyone interested in water resources systems analysis. The review reports are summaries of the state of the art. They are written mainly for beginners who are engaged in research on water resources systems analysis.

It is believed that the four reports mentioned above will be very useful to researchers, planners, and practicing engineers. Those who desire to do research on water resources systems can use these reports as a starting base in order to save individual effort which would be required to develop a list and review of the material. Planners for water resources development and managers of existing water projects will have a review of the techniques and the present and potential application of these techniques to the analysis of water resources plans so that they can see how the techniques have been or can be used and what techniques they should encourage their engineers to apply. The reports will be useful to the practicing engineer because they will provide him with the general approach of the techniques and then lead him to where he can find the details of a technique which he wishes to use.

Many persons assisted in the preparation of the four reports on 'Water Resources Systems Analysis.' The reports on Part I and Part III were produced with the support of OWRR Projects Nos. A-029-ILL and B-038-ILL, of which Ven Te Chow is Project Director and Principal Investigator, and Gonzalo Cortes-Rivera

and Sotirios J. Kareliotis are Research Assistants. The reports on Part II and Part IV were produced with the support of OWRR Project No. B-029-ILL, of which Ven Te Chow is Project Director and Co-Investigator, Dale D. Meredith is a Co-Investigator, and Eugene C. Cetwinski, James S. Windsor, and Chang-Lung Yin are Research Assistants. In addition, Manoutchehr Heidari, a graduate student in civil engineering, assisted in the preparation of the reports.

Ven Te Chow

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1. INTRODUCTION

1-1. General

The plan, or arrangement, of a water resources project may be called a system. Modern water resources projects often constitute very complex systems which may be created through different combinations of system units (reservoirs, canals, etc.), levels of outputs, and allocation of capacity of the units to various purposes (water supply, flood control, hydroelectric power, etc.) at different times. The objective of the system design is to select the combination of these variables that maximizes benefits in accordance with the requirements (constraints) of the design criteria. The constraints can be technical, economical, social or political and the benefits can be either real or implied.

Because of the unlimited, almost infinite, number of combinations that can be arranged in a multiunit-multipurpose water resources system, the optimal design cannot possibly be obtained by the conventional approach of incremental analysis [Federal Inter-Agency River Basin Commission, 1959]. By the techniques of operations research [McKean, 1958], however, it is possible to consider simultaneously a large number of alternative system designs and thereby isolate the optimal design.

The use of operations research in water resources planning and development has developed largely during the last 15 years. Special contributions to the knowledge are particularly due to a team effort at Harvard [Maass et al., 1962]. In this program, a model system is used as the test vehicle to analyze the river basin system. Two approaches of analysis are employed. One is to simulate the system on a high-speed digital computer and thereby select the best combination of variables by observing the response of the

simulated system to various alternative combinations. The other is to use simplified mathematical models which can be solved directly for the optimal design for relatively simple problems.

1-2. Nature of Programming Techniques

Programming problems deal with determining optimal allocations of limited resources to meet given objectives; more specifically, they deal with situations where a number of resources are available and are to be combined to yield some product or products. There are, however, certain restrictions on some or all of the resources.

The application of programming techniques begins by observing and formulating the problem. Next a scientific (typically mathematical) model that attempts to abstract the essence of the real problem is constructed. It is then hypothesized that this model is a sufficiently precise representation of the essential features of the situation so that the conclusions (solutions) obtained from the model are also valid for the real problem. This hypothesis is then modified and verified by suitable experimentation.

Programming techniques are applied in attempts to find the best or optimal solution to the problem under consideration. They make three contributions to the analysis of the problem. First, to use a programming technique one has to structure the real life situation into a mathematical model that abstracts the essential elements so that a solution relevant to the decision maker's objective can be sought. This involves looking at the problem in the context of the entire system. Second, in using a programming technique one has to explore the structure of such solutions and develop systematic procedures for obtaining them. And, third, having used a programming technique one has developed a solution that yields an optimal value of the system

measure of desirability (or possibly has compared alternative courses of actions by evaluating their measure of desirability).

1-3. Overview of the Report

Because the use of mathematical programming techniques in water resources system design is just beginning and no extensive applications or verifications have yet been made in practical problems, this report offers only a brief outline of the principles involved, not detailed problems. In order to provide the water resource planner with a reasonably adequate description of the modern mathematical programming techniques available to him, the techniques are explained and, then, examples of the problems which have been studied by these techniques are cited.

Chapter 2 is concerned with linear programming techniques. Non-linear programming techniques are discussed in Chapter 3 with the exception of dynamic programming which is covered in Chapter 4. Network theory, game theory, and simulation are outlined in Chapter 5.

2. LINEAR PROGRAMMING

2-1. Introduction

Linear programming deals with the problem of allocating limited resources among competing activities in an optimal manner. The problems which have been studied with linear programming include: (1) product mix, or how many of each of products A, B, and C to manufacture given limitations on man hours, machine hours, raw material, and other resources so as to minimize total production cost; (2) production scheduling, or what should the production rate be for a given commodity with a variable demand pattern so as to minimize costs associated with changing production levels from one period to the next and maintaining inventories; (3) machine loading problems, or how to assign production jobs to machines so as to minimize production costs; (4) 'diet' problems or what is the most economical mixture of raw materials which will result in a product with a desired composition, given the compositions and prices of the raw materials; and (5) transportation problems, or what is the minimum cost shipping pattern to transfer materials from various sources of fixed supply to points of specified demand.

Linear programming uses a mathematical model to describe the problem of concern. The major underlying assumptions of linear programming that limit its applicability are: (1) that all mathematical functions be linear functions and this in turn assumes that the measure of effectiveness and resource usage must be proportional to the level of each activity conducted individually; (2) that the total measure of effectiveness and each total resource usage resulting from the joint performance of the activities must equal the respective sums of these quantities resulting from each activity being conducted individually; (3) that it is permissible for an

optimal solution to contain fractional levels of resource usage; and (4) that all of the coefficients in the linear programming model are known constants.

2-2. The Linear Programming Model

The mathematical statement of a general linear programming problem is the following. Find the values of $x_1, x_2, x_3, \dots, x_n$ (called decision variables) which maximize (or minimize) the linear function (called the objective function)

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad (2.1)$$

subject to the following relationships (called constraints or restrictions)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \quad (\text{or } \geq b_1 \text{ or } = b_1) \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \quad (\text{or } \geq b_2 \text{ or } = b_2) \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \quad (\text{or } \geq b_m \text{ or } = b_m) \\ \text{all } x_j &\geq 0 \end{aligned} \quad (2.2)$$

Computational procedures (called algorithms) for solving such problems are outlined in nearly every textbook on linear programming [Dantzig, 1963; Gass, 1958; Hadley, 1962].

The decision variables, x_1, x_2, \dots, x_n , represent the levels of n competing activities. If each activity is the production of a certain product, then x_j would be the number of units of the j -th product to be produced during a given period of time. Z is the over-all measure of effectiveness, e.g., profit over the given time period. c_j is the increase in the objective function that would result from each unit increase in x_j . Each of

the first m linear inequalities corresponds to a restriction on the availability of one of the scarce resources. b_i is the amount of resource i available to the n activities. a_{ij} is the amount of resource i consumed by each unit of activity j . The restrictions that all x_j be greater than or equal to zero rule out the possibility of negative activity levels.

For convenience, the constraints are formulated such that $b_i \geq 0$ for all i . If in the original formulation any $b_i < 0$, that constraint can be multiplied by -1 . Given that each $b_i \geq 0$, the first step is to convert the constraints into a set of simultaneous linear equations. This is easily done by adding slack and surplus variables [Hadley, 1962]. If constraint i is of the form

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad (2.3)$$

then a slack variable, x_{n+i} , defined by

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j \quad (2.4)$$

is added such that

$$\sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i \quad (2.5)$$

For any set of x_j , $j = 1, 2, \dots, n$, which satisfy this constraint, $x_{n+i} \geq 0$. When a constraint is of the form

$$\sum_{j=1}^n a_{kj} x_j \geq b_k \quad (2.6)$$

then a surplus variable, x_{n+k} , defined by

$$x_{n+k} = \sum_{j=1}^n a_{kj} x_j - b_k \quad (2.7)$$

is added such that

$$\sum_{j=1}^n a_{kj}x_j - x_{n+k} = b_k \quad (2.8)$$

Then for any set of x_j which satisfy this constraint, $x_{n+k} \geq 0$. Therefore the constraints can be converted into a system of m linear equations in N unknowns, where $n \leq N \leq n+m$. The number of slack and surplus variables added will be $N-n$.

A feasible solution is a solution such that none of the decision variables is negative. A basic feasible solution is a feasible solution such that no more than m of the decision variables are positive. A degenerate solution is a basic feasible solution with fewer than m positive variables.

It can be proved [Hadley, 1962, 1964] that, given a linear programming problem in which degenerate basic feasible solutions do not occur, in which the optimum is unique, and in which additional basic feasible solutions can be formed, then the maximal solution must be a basic feasible solution. Therefore, if one is initially at a basic feasible solution, to find the optimal one need only move from one basic feasible solution to a better basic feasible solution. This is the central idea behind the simplex method of solving this type of problem [Dantzig, 1963; Gass, 1958; Hadley, 1962, 1964].

2-3. Duality Theory

Early in the development of linear programming it was discovered that every linear programming problem has associated with it another linear programming problem called the dual [Hadley, 1962].

Consider any linear programming problem in the form exhibited in Sec. 2-2. The primal problem is to find x_1, x_2, \dots, x_n in order to maximize

$$Z_x = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad (2.9)$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ &\vdots \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \\ \text{all } x_j &\geq 0 \end{aligned} \quad (2.10)$$

The corresponding dual problem is obtained by transposing the rows and columns of the coefficients in the constraint equations, transposing the coefficients of the objective function and the right-hand side of the constraint equations, reversing the inequalities, and minimizing instead of maximizing.

Therefore the dual problem is to find y_1, y_2, \dots, y_m in order to minimize

$$Z_y = b_1y_1 + b_2y_2 + \dots + b_my_m \quad (2.11)$$

subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m &\geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m &\geq c_2 \\ &\vdots \\ &\vdots \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m &\geq c_n \\ \text{all } y_i &\geq 0 \end{aligned} \quad (2.12)$$

The primal and dual problems have the following properties: (1) if either problem has an optimal solution so does the other, and furthermore, $\text{maximum } Z_x = \text{minimum } Z_y$, therefore the optimal values of the objective functions are equal; (2) the number of variables in the primal problem is equal to the number of constraints in the dual; (3) there is a dual variable associated with every primal constraint, and a primal variable associated with every dual constraint; and (4) by applying the simplex method to the dual problem, one can obtain a new algorithm, called the dual simplex method [Hadley, 1962], for solving the primal. When the dual is considered as a primal problem, the dual of the dual is the primal.

In addition to the procedures stated above, Lemke [1963] has enumerated the following rules to be followed when forming the dual problem from the primal.

(1) If the primal is a maximization problem, the dual will be a minimization problem, and vice versa. If the primal is of the maximization type, any \geq inequality in it should be converted to a \leq inequality before forming the dual. This is necessary to convert the problem into the form that agrees with the theorems from which the rules were drawn. If the primal is of the minimization type, any \leq inequality should be converted to a \geq inequality.

(2) If the j -th variable of the primal is unrestricted as to sign, the j -th constraint of the dual will be an equality.

(3) If the j -th variable of the primal is constrained to be non-negative, the j -th constraint of the dual will be an inequality of the type \geq if the primal is of the maximization type, and will be an inequality of the type \leq if the primal is of the minimization type. This should not be interpreted as implying that it is impossible for a minimization dual to contain a

\leq constraint, or a maximization dual to contain a \geq constraint. For, in either case, the i -th constraint may have to be multiplied through by -1 because c_i is negative and this will change the form of the constraint if it is an inequality.

(4) If the k -th constraint of the primal is an equality, the k -th variable of the dual will be unrestricted as to sign.

(5) If the k -th constraint of the primal is an inequality, the k -th variable of the dual must be constrained to be non-negative.

The concept of duality has an economic interpretation that is very useful. Let y_i^* denote the optimal value of the i -th dual variable. The optimal value of the dual variables, y_i^* ($i = 1, 2, \dots, m$), represents the shadow price¹ per unit of the respective resources. This allows one to answer the question of how much it is worth paying for additional resources, because y_i^* is the rate at which Z_x would increase (decrease) if the amount of resource i available were increased (decreased) over a certain range. This range is the range of b_i over which there is no change in which resources are the limiting resources in the optimal solution.

2-4. Integer Programming

In many problems, the decision variables make sense only if they have integer values. It is often necessary to assign men, machines, or vehicles to activities in integer quantities. The problem is called an all-integer problem if all variables are restricted to integer values and a mixed-integer-continuous variable problem if some variables are restricted to integers while others are allowed to vary continuously [Hadley, 1964].

¹ Other terms that are sometimes used include implicit value, incremental value, intrinsic value, internal price, efficiency price, and marginal value.

One approach to the integer linear programming problem is to begin with the original problem without the integer restriction and use the simplex method to find the optimal solution. If all of the variables have integer values, it is the optimal solution to the integer problem. If not, then the original linear programming problem is modified by adding a new constraint which eliminates some non-integer solutions, but which does not eliminate any feasible integer solutions. Since the new constraint has made the previous optimal solution infeasible, the dual simplex method can be applied to the modified problem to obtain a new optimal solution. If this solution is an integer solution, the problem is solved. If not, then another new constraint is added and the procedure is repeated. The optimal integer solution will be reached after enough new constraints have been added to eliminate all of the superior non-integer solutions. The key step in this procedure is the determination of the new constraint. Gomory [1963] and Llewellyn [1964] have presented methods for finding this constraint which satisfies a necessary, but not sufficient, condition for an integer solution.

Another approach to the integer linear programming problem is provided by the branch-and-bound technique [Lawler and Wood, 1966]. This is a type of enumeration procedure which partitions the set of all feasible solutions into several subsets, and then eliminates from further consideration those subsets which do not contain at least one solution which is better than the best feasible solution identified thus far. A subset is evaluated by the upper bound on the subset (the feasible solution in the subset with the maximum objective function) if the objective function is to be maximized, and by the lower bound on the subset (the feasible solution in the subset with the minimum objective function) if the objective function is to be minimized. From all of the remaining subsets, another one is selected for

further partitioning and testing. The procedure is repeated until a feasible solution is found such that the value of the objective function is no greater than the lower bound of any subset for a minimization problem, or if it is a maximization problem then the procedure is repeated until a feasible solution is found for which the value of the object function is greater than the upper bound for any subset.

2-5. Parametric Linear Programming

After a linear programming problem has been solved, it may be desirable to perform a sensitivity analysis or parameter study. A systematic study of changes in certain parameters in the linear programming model is the objective of parametric linear programming [Hadley, 1962].

Parametric linear programming allows one to determine how much the values of different parameters can change and the original optimal solution still remain optimal. It also allows one to investigate trade-offs between some of the parameter values. In many cases parametric linear programming will permit a new variable or a new constraint to be evaluated without solving the entire problem over again [Graves, 1963].

2-6. Linear Programming Under Uncertainty

In many cases of practical importance, water resource projects in particular, it turns out that some of the parameters appearing in the problem must be treated as random variables rather than as deterministic ones. There are various ways of formulating the problem of linear programming under uncertainty. These formulations can be classified into two types. One type, called stochastic programming, is where the constraints are required to hold with probability one [Madansky, 1963]. The second type, called chance-

constrained programming, is where feasible solutions are allowed to have a small probability of violating each constraint [Charnes and Cooper, 1959].

If some or all of the parameters are random variables, then the value of the objective function is also a random variable [Hadley, 1964]. Since it is meaningless to maximize a random variable, the usual approach is to reduce the problem to a deterministic case and solve it with the simplex method.

The three most usual methods of reducing the stochastic or chance-constrained programming problem to a deterministic linear programming problem are the expected value solution, the 'fat' solution, and the 'slack' solution. In the expected value solution, the technique is to replace the random elements by their expected values and proceed as in a deterministic linear programming problem [Dantzig, 1955]. In the 'fat' solution, the random elements are replaced by pessimistic estimates of the values of the random elements [Madansky, 1962, 1963]. In the 'slack' solution, the problem is recast into a two-stage problem where, in the second stage, one can compensate for 'inaccuracies' in the first stage activities [Dantzig, 1955]. Other possible methods have also been explored for chance-constrained programming [Charnes and Cooper, 1963].

If the coefficients in the i -th constraint must be treated as random variables, then this constraint disappears, and the relationship between the x_j 's and the density functions for the coefficients is used to determine the density function for s_i , the random variable representing the quantity of resource i used. The random variable s_i must be in the objective function to allow the computation of the expected cost incurred if more than the available resources are needed. Thus, making some of the parameters random variables appears to make the problem less constrained, which is true,

in a sense. However, the total effect when some of the coefficients are made random is that the problem becomes nonlinear and assumes a form for which, at present, there do not exist any general techniques for finding an optimal solution [Hadley, 1964].

2-7. Sequential Linear Programming

Sequential linear programming problems include those problems which involve the making of two or more decisions at different points in time, and which have the property that the latter decision(s) may be influenced by the previous decisions. If the latter decisions may also be influenced by some stochastic parameters whose values will actually have been observed before the decisions are made, the problem is one of stochastic-sequential linear programming. Many of these problems can be solved by their proper formulation as dynamic programming problems (Sec. 4).

Dantzig [1955] describes the expected value problem as a two-stage problem. He first chooses some values for the x_i , $i = 1, 2, \dots, n$, then observes the coefficients, and compensates for the discrepancies between the two sides of the constraints. His results show that the average of a number of minimum values of the objective function, derived for various values of the coefficients, cannot be lower than that minimum value derived from the average values of the coefficients. Basically, these models apply to situations where allocations in the first stage are made to meet an uncertain, but known distribution, of demands occurring in the second stage. These models can be extended for more than two stages.

Manne [1960a] has presented a technique for solving the sequential decision problem. The linear programming problem is formulated such that the unknowns, x_{ij} , $i = 1, 2, \dots, q$ and $j = 1, 2, \dots, r$, represent the joint

probabilities that if the system is in a state denoted by the subscript i that the decision variable will take a value denoted by subscript j . The states of a system are the various possible conditions in which the system might find itself. The time horizon is assumed to be infinite. The linear constraints are the requirements for statistical equilibrium, and the objective function to be minimized consists of the expected cost level corresponding to the equilibrium probabilities. This technique may prove to be an efficient alternative to the usual iterative method of solving dynamic programming problems [Manne, 1960a].

In the real world, most problems are sequential decision problems in some sense. Frequently, the random elements are important enough that they should also be considered to be stochastic programming in nature. It is only with the assumption that future decisions have a negligible effect on the current decision that they can be treated as non-sequential problems.

2-8. Applications of Linear Programming to Water Resources Systems

The method of linear programming has been applied to many types of water resources problems. The examples described below illustrate the type of problems that have been studied.

Mannos [1955] has presented a linear programming study of six multi-purpose reservoirs on the Missouri River. The objective was to maximize the hydroelectric energy that could be obtained from the entire system. The constraints on the system are the physical limitations on storages and releases.

Linear programming has been applied to the problem of water resources investment planning. Masse and Gilbrat [1957] used linear programming to determine the optimal allocation of investments in various types of power plants to

meet projected power requirements. Thermal power stations, hydroelectric stations with reservoirs, hydroelectric stations on rivers, power stations with sluice installations, and power stations operated by means of ocean tidal basins were included in the study. Linear programming has also been used in investment planning to determine the optimal timing and financing of sewage treatment facilities in response to a growing population [Lynn, 1964]. The type of treatment to be provided, the capacity required, and fund requirements are the constraints. The objective function is to minimize total cost of capacity enlargement, operation, and financing over the planning horizon. The decision variables or the outputs from the model are the sizes of treatment plants to be constructed in each time period.

Pavelis and Timmom [1960] combined benefit-cost analysis and linear programming to program small watershed development. Linear programming was used to maximize net benefits from the development, subject to specified amounts of land, labor or capital available for installing and maintaining the treatments. This approach required that the input-output data for each watershed-treatment activity be known.

A linear programming model was developed by Dorfman [1962] to determine the maximum net benefits from power supply and irrigation for a multi-purpose reservoir in a hypothetical river basin. The model was extended to include uncertainty.

Manne [1960a] formulated a stochastic model and optimized it by means of linear programming. He later developed a single reservoir, three-period model in which both current inflow and initial storage are assumed to be known [Manne, 1960b]. This model was used to develop product-mix curves for alternative operations of a reservoir which provides flood control, electric power, and irrigation. Thomas and Watermeyer [1962] expanded Manne's work to a

system of more than one reservoir and formulated a stochastic sequential model. Both inflow and storage were assumed variable in defining the initial stage of the reservoir system.

Thomas and Revelle [1966] employed linear programming to determine optimal operating policies for the High Aswan Dam. The objective function is to maximize the net benefits from hydropower production and irrigation uses. The decision variables are defined to be the amount of water to be released each month through the turbines and then to the headgates of the irrigation system. The main constraint is the quantity of water available. Other purposes of the reservoir such as flood control and navigation were not examined by the model.

Dracup [1966] formulated a mathematical model of a ground water-surface water system and analyzed the problem by using parametric linear programming. Changes in the cost coefficients were studied and the technique of cost ranging was used to indicate how trade-off amongst different sources of water could be mapped out to provide a pricing guide for a planning agent.

Linear programming has also been used in two-level optimization studies [Hall and Shephard, 1967]. A complex water resources system consisting of four rivers, ten reservoirs, associated power and pumping plants, and water supply aqueducts was decomposed into subsystems. Dynamic programming was used to optimize each subsystem for an assumed price schedule for system outputs. Then linear programming was used to optimize the entire system using the outputs from the subsystem optimization. The dual problem was then solved and, if the shadow prices were equal to the assumed prices in the subsystem optimization, the process was terminated. If the shadow prices and the assumed prices were not equal, then the shadow prices were used in the subsystem optimization process and the cycle was repeated until the shadow prices obtained

in the second-level optimization process were equal to the prices used in the subsystem optimization process.

Loucks [1968] has used stochastic linear programming models to define and evaluate various operating policies for several of the Finger Lakes in New York State.

The previous examples have been primarily concerned with the water quantity problem. Linear programming has also been used to study water quality problems. Lynn, Logan and Charnes [1962] demonstrated the use of linear programming to determine the optimal design for a sewage treatment plant.

Linear programming and integer programming models have been formulated by Deininger [1965] for a hypothetical river basin involving one water quality parameter.

Loucks, Revelle and Lynn [1967] and Revelle, Loucks and Lynn [1968] have used linear programming models to determine the least costly plan for waste treatment in a river basin. The decision variables are the degrees of BOD removal to be provided by each discharger for individual waste effluents. The constraints are that each discharger must provide partial or complete secondary treatment and that the dissolved oxygen concentration at any point in the stream must not go below a specified minimum value. A simplified version of the Willamette River in Oregon is studied in the later paper.

A linear programming model has been developed and applied to determine the minimum treatment cost to maintain at least a minimum dissolved oxygen concentration at all points in the Delaware estuary [Thomann, 1963 and 1965; Thomann and Sobel, 1964; Sobel, 1965]. Johnson [1967] used this model to evaluate four methods of allocating waste reductions among dischargers. The methods evaluated are uniform treatment for all users, minimum total cost of meeting the goal, a uniform price per unit of BOD discharged,

and an effluent charge varying with geographic area on each unit of BOD.

Liebman [1968] and Liebman and Marks [1968] have employed the branch-and-bound technique of linear programming to minimize the cost of waste treatment on a stream for zoned uniform treatment.

3. NONLINEAR PROGRAMMING

3-1. Introduction

Nonlinear programming models are similar to linear programming models except that the objective function and constraints are not required to be linear functions of the decision variables.

The general nonlinear programming problem is to find x_1, x_2, \dots, x_n so as to maximize

$$Z = f(x_1, x_2, \dots, x_n) \quad (3.1)$$

subject to

$$\begin{aligned} g_1(x_1, x_2, \dots, x_n) &\leq b_1 \quad (\text{or } \geq b_1 \text{ or } = b_1) \\ g_2(x_1, x_2, \dots, x_n) &\leq b_2 \quad (\text{or } \geq b_2 \text{ or } = b_2) \\ &\vdots \\ &\vdots \\ g_m(x_1, x_2, \dots, x_n) &\leq b_m \quad (\text{or } \geq b_m \text{ or } = b_m) \\ \text{all } x_j &\geq 0 \end{aligned} \quad (3.2)$$

where $f(x_1, x_2, \dots, x_n)$ and the $g_i(x_1, x_2, \dots, x_n)$ are given functions of the n decision variables. A minimization problem can easily be changed to a maximization problem by multiplying through by -1 and, therefore, these two can be considered equivalent.

The class of nonlinear programming problems which has been studied most extensively is that where the constraints are linear and the objective function is nonlinear. Even when attention is restricted to problems involving linear constraints, computational techniques for finding optimal solutions

have not been derived except in cases where the objective function has very special properties. For a variety of reasons, problems with nonlinear constraints tend to be much more difficult to solve than those with linear constraints. In most cases studied to date, problems with nonlinear constraints can most efficiently be solved by using one of the multi-variate search techniques.

3-2. The Kuhn-Tucker Conditions

The Kuhn-Tucker conditions describe the optimal solution to a nonlinear programming problem. These conditions were derived by Kuhn and Tucker [1951] and they are analogous to the conditions for an unconstrained optimal solution derived from classical calculus procedures. These conditions are given below.

If $f(x_1, x_2, \dots, x_n)$, $g_1(x_1, x_2, \dots, x_n)$, \dots , $g_m(x_1, x_2, \dots, x_n)$ are differentiable functions satisfying certain regularity conditions, then $(x_1^*, x_2^*, \dots, x_n^*)$ can be an optimal solution to the nonlinear programming problem only if there exist m numbers, $\lambda_1, \lambda_2, \dots, \lambda_m$ such that all of the following conditions are satisfied:

$$\text{If } x_j^* > 0, \text{ then } \frac{\partial f}{\partial x_j} - \sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \text{ at } x_j = x_j^* \quad (3.3)$$

for $j = 1, 2, \dots, n$

$$\text{If } x_j^* = 0, \text{ then } \frac{\partial f}{\partial x_j} - \sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial x_j} \leq 0 \text{ at } x_j = x_j^* \quad (3.4)$$

for $j = 1, 2, \dots, n$

$$\text{If } \lambda_i > 0, \text{ then } g_i(x_1^*, x_2^*, \dots, x_n^*) - b_i = 0 \quad (3.5)$$

for $i = 1, 2, \dots, m$

$$\text{If } \lambda_i = 0, \text{ then } g_i(x_1^*, x_2^*, \dots, x_n^*) - b_i \leq 0 \quad (3.6)$$

for $i = 1, 2, \dots, m$

$$x_j^* \geq 0 \quad \text{for } j = 1, 2, \dots, n \quad (3.7)$$

$$\lambda_i \geq 0 \quad \text{for } i = 1, 2, \dots, m \quad (3.8)$$

The λ_i are somewhat analogous to the dual variables of linear programming and they have a comparable economic interpretation. Actually the λ_i are generalized Lagrange multipliers. Equations (3.6) and (3.7) are included to help insure the feasibility of the solution. The other conditions eliminate many of the feasible solutions as possible candidates for the optimal solution. These conditions are only necessary, and not sufficient, for optimality. If certain additional convexity assumptions are satisfied, these conditions do become sufficient to guarantee optimality [Kuhn and Tucker, 1951].

It is difficult, and usually impossible to derive the optimal solution directly from the Kuhn-Tucker conditions [Hadley, 1964]. They provide clues for identifying the optimal solution, and they also may be used to determine if a proposed solution may be optimal.

3-3. Quadratic Programming

A quadratic programming problem is a nonlinear programming problem which has linear constraints and an objective function which is the sum of linear terms and quadratic terms. The problem is to find x_1, x_2, \dots, x_n so as to maximize

$$Z = \sum_{j=1}^n c_j x_j + \sum_{j=1}^n \sum_{k=1}^n d_{jk} x_j x_k \quad (3.9)$$

subject to

$$\sum_{j=1}^n a_{ij}x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \quad (3.10)$$

and

$$x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n \quad (3.11)$$

where the d_{jk} are given constants such that $d_{jk} = d_{kj}$.

Several solution procedures have been developed for the special case of the quadratic programming problem where the objective function is a concave function [Wolfe, 1963; Hadley, 1964]. The one described here was developed by Wolfe [1959] and has had considerable use.

The first step is to formulate the Kuhn-Tucker conditions for the problem. One form for expressing them for the quadratic case is as follows:

$$\sum_{k=1}^n d_{jk}x_k + \sum_{i=1}^m a_{ij}\lambda_i - y_j = c_j \quad \text{for } j = 1, 2, \dots, n \quad (3.12)$$

$$\sum_{j=1}^n a_{ij}x_j + y_{n+i} = b_i \quad \text{for } i = 1, 2, \dots, m \quad (3.13)$$

$$x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n \quad (3.14)$$

$$y_j \geq 0 \quad \text{for } j = 1, 2, \dots, n+m \quad (3.15)$$

$$\lambda_i \geq 0 \quad \text{for } i = 1, 2, \dots, m \quad (3.16)$$

$$\lambda_i y_{n+i} = 0 \quad \text{for } i = 1, 2, \dots, m \quad (3.17)$$

$$x_j y_j = 0 \quad \text{for } j = 1, 2, \dots, n \quad (3.18)$$

where y_j ($j = 1, 2, \dots, n+m$) are slack variables. Except for equations (3.17) and (3.18), these conditions are nothing more than linear programming constraints involving $2(n+m)$ variables. Furthermore equations (3.17) and (3.18) simply say that it is not permissible for both x_j and y_j ($j = 1, 2, \dots, n$)

or both λ_i and y_{n+i} ($i = 1, 2, \dots, m$) to be basic variables when considering basic feasible solutions. Therefore, the problem reduces to finding an initial basic feasible solution to any linear programming problem having these constraints. Since most or all of the c_j are usually positive, it is not obvious what the initial basic variables should be for equations (3.12). The standard linear programming procedure to overcome this difficulty is to introduce artificial variables which are eventually forced to be equal to zero. Therefore let z_j ($j = 1, 2, \dots, n$) be these artificial variables subject to

$$z_j \geq 0 \quad \text{for } j = 1, 2, \dots, n \quad (3.19)$$

Then equations (3.12) become

$$\sum_{k=1}^n d_{jk} x_k + \sum_{i=1}^m a_{ij} \lambda_i - y_j + c_j z_j = c_j \quad \text{for } j = 1, 2, \dots, n \quad (3.20)$$

Equations (3.13) and (3.20) now provide an artificial initial basic feasible solution. The optimal feasible solution must satisfy equation (3.19).

Therefore the next step is to start with the initial basic feasible solution given above and apply a modification of the simplex method so as to minimize

$$Z' = \sum_{j=1}^n z_j \quad (3.21)$$

subject to equations (3.20), (3.13), (3.14), (3.15), (3.16), and (3.19).

The modification to the simplex method is that equations (3.17) and (3.18) must also be satisfied. When the optimal solution,

$(x_1^*, \dots, x_n^*, \lambda_1^*, \dots, \lambda_m^*, y_1^*, \dots, y_{n+m}^*, z_1 = 0, \dots, z_n = 0)$ is obtained for this problem, (x_1^*, \dots, x_n^*) is the optimal solution for the original quadratic programming problem.

The computation required to solve a quadratic programming problem involving m constraints and n variables is then roughly the same as that involved in the solution of a linear programming problem with $n + m$ constraints.

3-4. Geometric Programming

Geometric programming was designed for problems in which the constraints are nonlinear or the objective function is of more than second degree. The technique was first used by Zener [Duffin, 1962], and further developed by Duffin, Peterson, and Zener [1967]. In its present form, the geometric programming algorithm can handle a large class of problems without the necessity of using questionable linear or quadratic approximations.

The general unconstrained geometric programming problem is to minimize

$$Z = c_1 x_1^{a_{11}} x_2^{a_{12}} \dots x_n^{a_{1n}} + c_2 x_1^{a_{21}} x_2^{a_{22}} \dots x_n^{a_{2n}} + \dots + c_m x_1^{a_{m1}} x_2^{a_{m2}} \dots x_n^{a_{mn}} \quad (3.22)$$

Equation (3.22) can be rewritten as

$$Z = \sum_{i=1}^m c_i \prod_{j=1}^n x_j^{a_{ij}} \quad (3.23)$$

where there are n variables and m terms in the equation. As in the classical optimization method, the first partial derivatives of equation (3.23) are set to zero. Therefore,

$$\left(\frac{\partial Z}{\partial x_k} \right)^* = \frac{1}{x_k^*} \sum_{i=1}^m a_{ik} c_i \prod_{j=1}^n (x_j^*)^{a_{ij}} = 0 \quad \text{for } k = 1, 2, \dots, n \quad (3.24)$$

where the asterisk (*) superscript denotes the value of the variable which causes equation (3.24) to be equal to zero. Now define the optimal weights, w_i , as

$$w_i = \frac{1}{z^*} \left\{ c_i \prod_{j=1}^n (x_j^*)^{a_{ij}} \right\} \quad \text{for } i = 1, 2, \dots, m \quad (3.25)$$

If z^* and all of the x_k^* 's are non-zero, then equations (3.24) can be substituted into equation (3.25) to obtain equation (3.26).

$$\sum_{i=1}^m a_{ij} w_i = 0 \quad \text{for } j = 1, 2, \dots, n \quad (3.26)$$

In addition these weights must sum to unity so that

$$\sum_{i=1}^m w_i = 1 \quad (3.27)$$

Equations (3.26) and (3.27) are linear in the weights and do not depend upon the coefficients, c_i .

If the number of terms on the right side of the equal sign in equation (3.23) exceed the number of variables of exactly one, then there are exactly as many linear equations in equations (3.26) and (3.27) as there are weights, w_i , so that these equations can be solved to determine the value for the weights. Since equation (3.27) must be valid, it can be proved that

$$z^* = \prod_{i=1}^n (c_i / w_i)^{w_i} \quad (3.28)$$

and hence the minimum value of the objective function can be determined without knowledge of the optimal values of the decision variables. Then the optimal values of the decision variables can be determined by substituting the known values of z^* and w_i into equation (3.25) and solving for the x_j^* .

The difference between the number of terms on the right side of the equal sign in the objective function and the number of independent linear equations is referred to as the degrees of freedom or the degrees of difficulty and is equal to $m - (n+1)$. If the degrees of freedom is greater than zero,

an infinite number of solutions is possible for the weights.

Geometric programming has been generalized to allow the solution of problems with a few degrees of difficulty, to allow the use of negative coefficients in both the objective function and the constraints, and also to permit reversed inequality constraints [Duffin, Peterson, and Zener, 1967].

Geometric programming is a technique whereby highly nonlinear systems can be analyzed. The main theme of the geometric programming approach requires that the engineering characteristics of the optimization problem be expressed quantitatively as generalized polynomials in the decision variables.

3-5. Search Techniques

The optimization problem is one of determining the value of the decision variables such that the objective function is optimized. For centuries, the problem of optimization has been attacked by simply defining the criteria for optimality, and searching for the optimum in some manner. The previous sections have been concerned with optimization techniques which require the solution to equations to decide the optimum, rather than searching for the optimum. These techniques work, basically because the roots of the equations also represent the location of the optimum. Many times these techniques can not be applied and one must use a search technique to find the optimum.

Search plans can be categorized as to the type of problem being searched. A univariate search plan searches a problem involving only one decision variable, while a multi-variate search must consider several decision variables and their effects on optimality. A search plan can also be an unconstrained search or a constrained search. An unconstrained search has no

constraining functions which limit the region of feasibility, and a constrained search must satisfy certain constraints which limit the region of feasibility to be considered for optimality.

Several terms need to be defined before a comprehensive discussion of search techniques can be made. Unimodality is a property which implies that the function Z of a variable x has only one 'hump' in the interval to be explored. This basically means that a local maxima is also the global maxima. The region to be explored is called the 'interval of uncertainty' and the region left after n trials have been performed is the interval of uncertainty after n experiments and is denoted by L_n . After a series of experiments has been carried out and the results have been measured, L_n will be a good indication of how effective the search has been. L_n is the longest possible length of the final interval of uncertainty and is therefore unique. Because L_n is deliberately chosen to correspond to the worst possible outcome, it is completely free of troublesome dependence on experimental results. Instead of trying to guess which interval of uncertainty will remain, one simply deals with the longest interval of uncertainty that might turn up if his luck is bad. Thus by confining his attention to an extreme case, he actually obtains an a priori measure of search effectiveness; pessimistic as it may be. The tolerance, ϵ , is the degree of precision desired and is determined prior to attempting an experiment. It is dependent upon the extent to which the experimenter is physically capable of differentiating between two outcomes of the experiments.

3-5-1. Univariate Search Schemes

Search plans fall into two mutually exclusive categories: simultaneous and sequential. Plans specifying the location of every experiment before any results are known as simultaneous search plans. Those permitting the

experimenter to base future experiments on past outcomes are considered as sequential search plans.

3-5-1-1. Simultaneous Search

When all experiments must be run at the same time, it is necessary to use a simultaneous search plan. Naturally, simultaneous schemes are much less effective than sequential plans in which the locations of later experiments can be based on the results of earlier ones. But there are situations where the experimenter is forced to use a simultaneous plan.

In a simultaneous search an odd number of trials (experiments) are of virtually no use to the experimenter [Wilde, 1964]. Therefore the most effective plan involves the use of pairs of trials, known as a 'search by uniform pairs'. The strategy is to space pairs of trials, a distance ϵ apart, evenly in the original interval of uncertainty to be examined.

3-5-1-2. Sequential Search

A sequential search allows the investigator to run his experiments one after the other and use information from earlier experiments to decide where to locate later ones.

3-5-1-2-1. Dichotomous Search

When there are only two trials, the best thing to do is to place both trials at the center of the interval and as close together as possible. This leaves an interval of uncertainty of length $L_0(1/2 + \epsilon/2)$ where L_0 is the original interval of uncertainty. If the third and fourth trials are placed in the middle of the remaining interval, the interval of uncertainty would be given by $L_0(1/4 + \epsilon/4)$. In general, after n trials ($n = \text{an even number}$) one

can locate the optimum within an interval of length $L_0[2^{-n/2} + (1-2^{-n/2})\epsilon]$.

This is known as the dichotomous search plan [Wilde, 1964] and its effectiveness grows exponentially with n , while that of a search by uniform pairs increases only in direct proportion to the number of trials.

3-5-1-2-2. Fibonacci Search

The Fibonacci search technique relates the lengths of successive intervals of uncertainty [Wilde, 1964].

In order to facilitate a compact general relationship for these lengths, a sequence of Fibonacci numbers, F_k , are defined as:

$$\begin{aligned} F_0 &= F_1 = 1 \\ F_k &= F_{k-1} + F_{k-2} \quad k = 2, 3, \dots, m \end{aligned} \quad (3.29)$$

In terms of Fibonacci numbers, the remaining interval of uncertainty may be expressed as

$$L_{n-k} = F_{k+1}L_n - F_{k-1}\epsilon \quad (3.30)$$

If the length of the original interval of uncertainty is L_1 , then $n-k = 1$ and equation (3.30) can be written as

$$L_1 = F_n L_n - F_{n-2}\epsilon \quad (3.31)$$

from which the fraction L_n of the original interval of uncertainty that remains after n sequential experiments can be obtained. Therefore,

$$L_n = L_1/F_n + F_{n-2}\epsilon/F_n \quad (3.32)$$

Once a Fibonacci search has been begun, it is easy to decide what to do at each stage. To continue the search, all one needs to do is locate

the next trial symmetrically with respect to the one already in the interval. The first experiment must be placed L_2 units from one end of the original interval of uncertainty. Because of symmetry, it does not matter which end. To obtain L_2 , one uses equation (3.30) with $n-k = 2$ such that

$$L_2 = F_{n-1}L_n - F_{n-3}\epsilon \quad (3.33)$$

If equation (3.32) is substituted into equation (3.33) to eliminate L_n and the result simplified, one obtains

$$L_2 = F_{n-1}/F_n + (F_{n-1}F_{n-2} - F_nF_{n-3})\epsilon/F_n \quad (3.34)$$

Thus, the interval of uncertainty can be reduced to less than one per cent of its original length after only eleven sequential trials.

In order to apply this technique one must know the length of the original interval of uncertainty and how many trials are to be made. This last requirement however, can be circumvented in most cases by assuming n to be sufficiently large. A value for n of approximately 10 is ordinarily sufficient.

3-5-1-2-3. Golden Section Search

Often an experimenter begins searching for an optimum without advance knowledge of exactly how many trials to use. He simply keeps experimenting until the criterion becomes satisfied. Although the dichotomous search plan can be utilized, there exists another technique which, while nearly as effective as the Fibonacci method, is completely independent of the number of experiments available. This technique is known as the search by golden section [Hadley, 1964]. It calls for the first trial to be placed a distance of $0.62 L_0$ (L_0 is the original interval of uncertainty) from one end of the interval and the second, $0.62 L_0$ from the other end. After eliminating the

infeasible segment discovered by the first two trials, the remaining interval will contain one of the previous trials. To continue the search, one merely places the next experiments symmetrically in the remaining intervals. After n experiments, the interval L_n remaining is given by,

$$L_n = 1/\tau^{n-1} \quad (3.35)$$

where $\tau = 1.62$.

3-5-1-2-4. Lattice Search

It is not unusual to encounter problems in which the independent variable, x , cannot vary continuously within the given interval of uncertainty, but instead is confined to a finite number of points. The important characteristic of these lattice search problems is that the number of points be finite and arrangeable in some order that will make the criterion of effectiveness unimodal [Wilde, 1964]. If the number of points happens to be exactly one less than a Fibonacci number, the most efficient method is to use, in a straightforward fashion, the Fibonacci technique. In other cases, it is most efficient to add fictitious points to the original set until the total is one less than a Fibonacci number and proceed with the Fibonacci search.

3-5-1-2-5. Randomization Search

In the randomization search, certain decisions are decided by chance; the flip of a coin or the use of a table of random numbers (Monte Carlo techniques) [Hadley, 1964]. The use of a random device for decision-making is theoretically justified in the development of mixed strategies in game theory. The technique simply associates a probability with a random outcome, and a corresponding decision rule to that random outcome. For example, if there are

six feasible locations in a lattice search, one location can be associated with each outcome of throwing an unbiased die; each of these outcomes, of course with an associated probability of 1/6. The random device is implemented, the die thrown, and the decision is based upon the outcome.

3-5-2. Multivariate Search Plans

At first glance, one might think that the difference between multivariate search problems and the single variable ones already analyzed is only one of degree and that one could extend single variable methods. Unfortunately this is not true since multivariate problems have a structure entirely different from that of single variable problems. Bellman [1961] refers to the difficulties engendered by the differences in multivariability as 'the curse of dimensionality.'

Before an understanding can be expected in this type of problem, one must be acquainted with the concept of the gradient vector in a multi-dimensional space. The gradient vector can be defined mathematically as

$$\nabla f(x'_1, x'_2, \dots, x'_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) \quad (3.36)$$

$$\text{at } (x_1, x_2, \dots, x_n) = (x'_1, x'_2, \dots, x'_n)$$

Some important properties of the gradient vector include the following:

(1) at any point the gradient vector is normal to the level surface passing through that point; (2) at any point the gradient vector points in the direction in which the function is increasing most rapidly; and (3) at any point the negative of the gradient vector points in the direction in which the function is most rapidly decreasing.

The n-dimension problems are considered here in terms of unconstrained and constrained problems.

3-5-2-1. Unconstrained Problems

The unconstrained optimization problem is to maximize

$$Z = f(x_1, x_2, \dots, x_n) \quad (3.37)$$

for a given starting point $(x_1^0, x_2^0, \dots, x_n^0)$. The initial starting point may be obtained in many ways: an approximation guess, an exercise of engineering judgement, or a mathematically calculated starting point.

The steepest ascent method is the only procedure that is discussed here for unconstrained problems. This iterative technique derives its name from the fact that the sampling moves sequentially from lower to higher values of the objective function and it does this along the steepest and hence the shortest path to the optimum [Cochran and Cox, 1957]. The values of $\partial f / \partial x_j$ ($j = 1, 2, \dots, n$) in equation (3.36) define the rate of change of the objective function relative to the individual x_j . Therefore, moving from the starting x_j^0 ($j = 1, 2, \dots, n$) to the revised value x_j^1 requires that the change in each variable be made proportional to the associated rate of improvement $\partial f / \partial x_j$. Thus the revised value for x_j^1 is computed as

$$x_j^1 = x_j^0 + d \left(\frac{\partial f}{\partial x_j} \right) \quad (3.38)$$

where d is a constant of proportionality. One way to choose d is to define a maximum absolute value of change for any variable. Let this maximum value of change be k . Then,

$$\max_j |x_j^1 - x_j^0| = k \quad (3.39)$$

and

$$d = k / \max_j | \partial f / \partial x_j | \quad (3.40)$$

The units of x_j must be the same for all j for this method to be valid.

3-5-2-2. Constrained Problems

The constrained optimization problem is stated by equations (3.1) and (3.2). The number of constraints, m , is less than the number of variables, n . This defines the degrees of freedom, $n-m$.

3-5-2-2-1. Lagrangian Multiplier Technique

If equations (3.2) are equality constraints, then the problem can be reformulated so as to maximize the Lagrangian, L , which is defined as

$$L = f(x_1, x_2, \dots, x_n) + \sum_{i=1}^m \lambda_i [g(x_1, x_2, \dots, x_n) - b_i] \quad (3.41)$$

where λ_i ($i = 1, 2, \dots, m$) are the Lagrangian multipliers. The necessary conditions for L to be a maximum are

$$\begin{aligned} \frac{\partial L}{\partial x_j} &= 0 & j &= 1, 2, \dots, n \\ \frac{\partial L}{\partial \lambda_i} &= 0 & i &= 1, 2, \dots, m \end{aligned} \quad (3.42)$$

This results in $n + m$ equations with $n + m$ unknowns. The procedure for the case of inequality constraints is similar, although, somewhat more complicated [Hadley, 1964]. Lagrange multipliers are used in the derivation of the Kuhn-Tucker conditions for an optimal [Sec. 3-2.].

3-5-2-2-2. Sequential Unconstrained Minimization Technique

The sequential unconstrained minimization technique solves the constrained minimization problem by solving a series of unconstrained minimiza-

tion problems [Fiacco and McCormick, 1964a, 1964b]. This is facilitated by constructing a new objective function which incorporates the constraints of the original problem. The general problem is to minimize

$$Z = f(x_1, x_2, \dots, x_n) \quad (3.43)$$

subject to

$$\begin{aligned} g(x_1, x_2, \dots, x_n) &\geq b_i & i = 1, 2, \dots, m \\ \text{all } x_j &\geq 0 \end{aligned} \quad (3.44)$$

Let

$$h_i(x_1, x_2, \dots, x_n) = \begin{cases} b_i - g_i(x_1, x_2, \dots, x_n) & \text{if } i = 1, 2, \dots, m \\ x_i - m & \text{if } i = m+1, m+2, \dots, m+n \end{cases} \quad (3.45)$$

The problem is then to minimize equation (3.43) subject to

$$h_i(x_1, x_2, \dots, x_n) \geq 0 \quad i = 1, 2, \dots, m+n \quad (3.46)$$

Now define a new function, P, as

$$P = f(x_1, x_2, \dots, x_n) + r \sum_{i=1}^{m+n} \frac{1}{h_i(x_1, x_2, \dots, x_n)} \quad (3.47)$$

P is the function that is to be minimized for different values of r, and as r becomes negligibly small the minimum value of P approaches the minimum value for Z.

The theoretical validation of this procedure and a detailed algorithm for its implementation have been presented [Fiacco and McCormick, 1964a; 1964b].

3-6. Applications of Nonlinear Programming to Water Resources Systems

Nonlinear programming has not enjoyed the popularity that linear programming has in water resources system analysis. This is partially due to the fact that currently available mathematical theory and numerical techniques are not sufficient to generate solutions to all kinds of nonlinear programming problems. Also many nonlinear problems have been analyzed using dynamic programming [Sec. 4].

One approach to solving nonlinear problems is to approximate nonlinear functions with linear functions and to determine an approximate solution by linear programming. This is the approach Kerri [1966, 1967] used to determine the degree of treatment required to maintain at least a given minimum dissolved oxygen concentration in a river such that the total treatment costs are minimized. The model is applied to a group of 5 industrial waste dischargers on the Willamette River in Oregon.

The use of nonlinear programming techniques in water treatment problems has also been demonstrated by Thomas and Burden [1963, Chap. 5]. They use hypothetical data to design the capacity for a water treatment plant such that the net benefits from the plant are to be maximized.

Lynn [1966] demonstrates a quadratic programming approach to the selection of a least cost pumping schedule for withdrawing a community water supply from a group of wells.

Some of the search techniques have been used with simulation in order to identify the optimum value for the objective function [Maass et al., 1962].

4. DYNAMIC PROGRAMMING

4-1. Introduction

Dynamic programming is a mathematical technique which is useful in solving sequential decision problems [Bellman, 1957, 1961; Bellman and Dreyfus, 1962]. A sequential decision problem is a problem in which a sequence of interrelated decisions, termed a policy, must be made. Dynamic programming is an efficient enumeration procedure for determining the combination of decisions which maximizes overall effectiveness as measured by some criterion (objective) function.

There exists no standard mathematical formulation for dynamic programming problems. It is an approach oriented technique, and the particular equations used must be developed to fit each individual problem. Therefore, an understanding of the general structure of dynamic programming problems is required to recognize when a problem can be solved by dynamic programming procedures, and how it should be done.

There are four features which characterize the problems to which the dynamic programming approach can be applied. First, the problem must be one which can be divided into stages with a decision required at each stage. The stages may represent different points in space, as for example in selecting a route for a new pipeline, or they may represent different points in time, as for example in determining the optimal releases from a reservoir. Second, each stage of the problem must have a finite number of states associated with it. The states describe the possible conditions in which the system might find itself at that stage of the problem. In reservoir operation studies, the states may represent the amount of water stored in the reservoir at that stage. Third, the effect of a decision at each stage of the problem is to transform the current state of the system into a state associated with the next

stage. The decision may represent how much water to release from the reservoir at the current time, and this decision will transform the amount of water stored in the reservoir from the current amount to a new amount for the next stage. Associated with each potential state transformation is a benefit or cost which indicates the effectiveness of the transformation. Fourth, for a given current state and stage of the problem the optimal sequence of decisions is independent of the decisions made in previous stages.

Dynamic programming is a simple procedure from the computational point of view, and one which can treat nonconvex, nonlinear, discontinuous objective and constraint functions. It also permits the same analytic method to be used for the treatment of both stochastic and deterministic problems. The mathematical principles underlying dynamic programming are briefly outlined below along with an indication of the types of water resources problems to which it has been applied.

4-2. Recursive Equation

The dynamic programming approach can best be explained by referring to an example. Assume that a reservoir is to be operated for N months such that a release is to be made from the reservoir each month, that the benefits to be obtained each month are some functional value of that release, and that the objective is to maximize the value of the benefits. The schematic representation (functional diagram) of this multistage process is shown in Figure 4-1. The stages, which are analogous to the points in time at which the release decisions have to be made, are indicated by squares. These squares are numbered in reverse order by computational convention. The states are the inputs to and outputs from the stages and are analogous to the amounts of storage in the reservoir before and after the releases are made. States are

illustrated on the functional diagram by horizontally directed arrows and are labeled s_i and \tilde{s}_i to indicate, respectively, the input and output states from the i -th stage. In a more complicated system of several reservoirs s_i could be conceived as a vector whose components represent the storages in individual reservoirs. The return r_i is a measure of the utility derived from releasing a quantity of water d_i in the i -th stage, where d_i is called the decision at stage i , the q_i is the inflow to the reservoir in the i -th stage, and e_i is the reservoir losses in the i -th stage. Decisions, returns, inflows, and losses are indicated as vertically directed arrows on the functional diagram. A transformation function, T_i , defines the way in which an input state is transformed into an output state by the decision variable. The output state is then given by equation (4.1). If the stage is one month,

$$\tilde{s}_i = T_i(s_i, d_i) \quad i = 1, 2, \dots, N \quad (4.1)$$

then the output state is equal to the input state plus any inflow to the reservoir during the month minus the sum of losses from the reservoir due to evaporation or overflow during the month and any release made such that the transition function is given by

$$T_i(s_i, d_i) = s_i + q_i - e_i - d_i \quad i = 1, 2, \dots, N \quad (4.2)$$

The manner in which stages are interconnected is described by incidence identities. The incidence identities used in serial systems are given in equation (4.3). These identities

$$\tilde{s}_{i+1} = s_i \quad i = 1, 2, \dots, N-1 \quad (4.3)$$

simply state that the output from each stage forms the input to the next

succeeding stage.

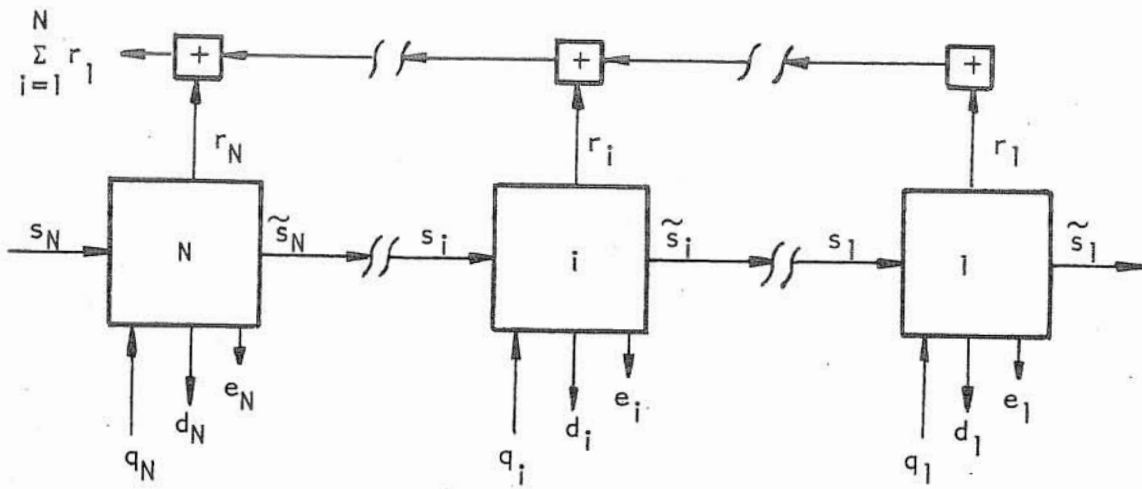


FIGURE 4-1. Functional Diagram for N-Stage Serial System

The return from the operation of each stage will be expressed as a function of the storage at the beginning of the stage and the quantity of water released at that stage such that

$$r_i = r_i(s_i, d_i) \quad (4.4)$$

At each stage a value of the decision variable d_i may be selected from a set of allowable values in the ranges

$$d_i \leq s_i + q_i - e_i \quad (4.5)$$

and

$$0 \leq d_i \leq Q_{\max} \quad (4.6)$$

where Q_{\max} is the maximum allowable release during any month. That is, the system output at any stage is constrained either by a design variable or by the

amount of water in the reservoir. Similarly, the active storage in the system will be bounded by some upper limit s_{\max} which depends on the scale of development. Therefore, s_i will lie in the set

$$0 \leq s_i \leq s_{\max} \quad (4.7)$$

The serial optimization problem is to find the policy, or set of decision variables $(d_N, d_{N-1}, \dots, d_1)$, which maximizes the total return R from the project. Therefore, the multistage decision problem may be defined as

$$\text{Max. } R = f_N(s_N) = \text{Max. } \sum_{i=1}^N r_i(s_i, d_i) \quad (4.8)$$

subject to equations (4.5), (4.6), and (4.7).

The problem, therefore, is to devise an efficient method which will produce the maximum value of R after examining all of the feasible combinations of storage, s_i , and outflow, d_i , subject to the above constraints. This is accomplished by dynamic programming which transforms the N -decision one-state initial value optimization problem into a set of N one-decision one state problems. The basic recursive (functional) equation is derived as follows:

$$f_N(s_N) = \text{Max.}_{d_N, d_{N-1}, \dots, d_1} \sum_{i=1}^N r_i(s_i, d_i) \quad (4.9a)$$

$$= \text{Max.}_{d_N, d_{N-1}, \dots, d_1} \{r_N(s_N, d_N) + r_{N-1}(s_{N-1}, d_{N-1}) + \dots + r_1(s_1, d_1)\} \quad (4.9b)$$

$$= \text{Max.}_{d_N} \text{Max.}_{d_{N-1}, d_{N-2}, \dots, d_1} \{r_N(s_N, d_N) + r_{N-1}(s_{N-1}, d_{N-1}) + \dots + r_1(s_1, d_1)\} \quad (4.9c)$$

$$= \text{Max.}_{d_N} \{r_N(s_N, d_N)\} + \text{Max.}_{d_{N-1}, d_{N-2}, \dots, d_1} \{r_{N-1}(s_{N-1}, d_{N-1}) + \dots + r_1(s_1, d_1)\} \quad (4.9d)$$

Since the definition of the maximum return with respect to the decision variables (d_{N-1}, \dots, d_1) is similar to the original definition of $f_N(s_N)$ if the limit $N-1$ is replaced by N , equation (4.9d) may be rewritten as

$$f_N(s_N) = \text{Max}_{d_N} \{r_N(s_N, d_N) + f_{N-1}(s_{N-1})\} \quad (4.10)$$

or in general form as

$$f_i(s_i) = \text{Max}_{d_i} \{r_i(s_i, d_i) + f_{i-1}(s_{i-1})\}; \quad i=1, 2, \dots, N \quad (4.11)$$

subject to equations (4.5), (4.6), and (4.7) and f_0 is defined to be zero.

Equation (4.11) is the recursive equation of dynamic programming. In general terms, it states that the total maximum return from the system in state s_i at the i -th stage is obtained by adding the return of the i -th month, produced by decision d_i , to the return from all future stages using the resulting state and an optimal policy. The first term in equation (4.11), $r_i(d_i, s_i)$, does not depend on the set of decisions (d_{i-1}, \dots, d_1) , while the return $f_{i-1}(s_{i-1})$ is affected by the entire set of decisions.

The recursive equation is the mathematical interpretation of Bellman's principle of optimality, and shows that the optimal return $f_i(s_i)$ can be obtained from the optimal return $f_{i-1}(s_{i-1})$ by a one state, one decision optimization problem. The recursive equation therefore permits one to solve a sequence of related problems one at a time starting at the last stage and progressing backwards in time to the initial stage.

Since the intermediate state s_i is an output from a previous stage, one cannot know which of its values is optimal until the entire problem has been solved. For this reason, an optimal decision d_i must be determined for all feasible values of s_i , and these values must be stored in order to generate

an optimal policy after termination of the backward algorithm.

4-3. Algorithm

Since dynamic programming is an iterative procedure a relatively small number of computer instructions is required. Further, constraints imposed on the system due to flow and water level limitations reduce the number of feasible policies and therefore the time required to establish the optimal policy.

The computational procedure of dynamic programming only considers a number of feasible storages and flow releases in the search for an optimal policy. A decision, therefore, has to be made with respect to the intervals Δs between successive states and Δd between successive decisions. These are known as the grid spacings.

If a coarse grid is selected, the number of feasible values of the state and decision variables is reduced. This results in a reduction in computer time, but only at the expense of accuracy, since there is a greater chance of missing the true optimum of the system.

An alternative method is to solve the problem in a number of steps beginning with only a few widely spaced values for each variable. On the basis of this solution a number of the grid points may be eliminated. A finer grid is then selected to obtain a closer approximation of the true optimum using the smaller feasible region.

Considering the case of the backward algorithm the computation proceeds in a direction opposite to real time, starting with the last month of the planning period. At each stage of the process there are a number of discrete feasible storages and for each storage a number of discrete feasible releases which are bounded by the constraints on the system. The recurrence equation has therefore to be solved for a set of discrete points $\{(s_k, d_j)\}$.

where $k = 1, \dots, K$ and $j = 1, \dots, J$ at each stage i in the process. This yields a set of optimal releases for all storage values.

Thus for the first stage (i.e., last time period), the result of the computation process is a table of initial storages versus returns for optimal use of water in this period. Since the storage at the end of this stage is assumed and since there are no future returns to be considered, the course of action for this period is fairly simple. That is, the optimal policy will be the one that causes the final assumed end storage to be obtained for all the discrete values of initial storage for this stage. This computational procedure is repeated for the next stage. In this case, however, the returns obtained from releasing water in the second stage is balanced against the value of initial storage obtained from the first stage. The recursive nature of the equation should now be clear. As each stage is added, an optimal allocation is made between the newly added stage and the optimal policy in all the previous stages. This adding on procedure is continued until the entire planning period has been considered. The outcome is a matrix which gives the optimal policies and benefits for the system.

It is not until the optimal return has been calculated for the entire system that the optimum values for the intermediate stages can be determined. The second part of the algorithm, therefore, is to seek out the optimum set of releases which will maximize the return obtained from the backward algorithm. This is accomplished by moving in the direction of real time from stage N to 1. Thus at the start of the planning period the optimal decision and output are selected from the matrix calculated by the backward algorithm knowing the initial storage value. The storage level at the start of the next succeeding month is determined using the transition function, incidence identities, and the given values of storage s_N , flow release d_N , inflow q_N , and losses e_N for

month N. Referring again to the matrix obtained by the backward algorithm the optimal flow release and output may be selected for the second month in the planning period. This procedure is repeated at each successive stage in the planning period. The result of the forward algorithm is therefore a complete set of releases which maximizes the system output for the entire planning period.

4-4. A Worked Example

A simple, hypothetical worked example will be used to acquaint the reader with the details of the recursive equation and algorithm discussed previously.

Consider a pipe distribution system which supplies water to 3 separate outlets as shown in Figure 4-2. Assume that the benefits at each outlet depend only on the amount of water supplied at that outlet. The problem is to determine the allocation of water which yields the maximum return from the system assuming that the maximum amount of water available is 3 acre-feet.

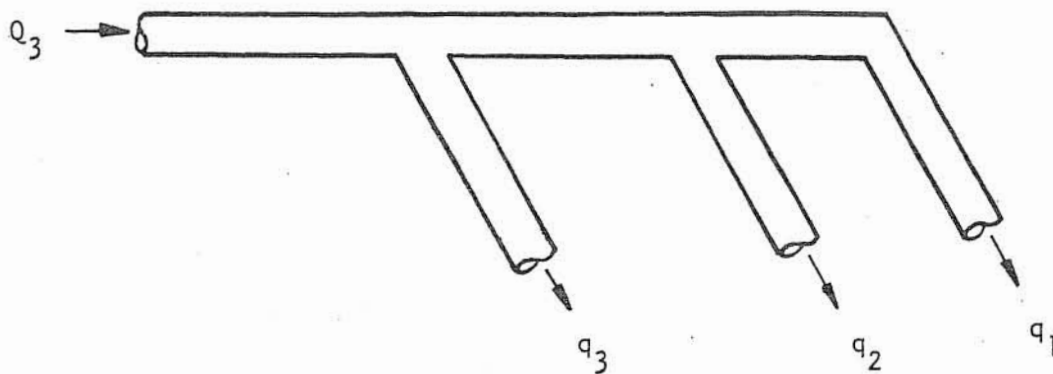


FIGURE 4-2. Pipe Distribution System which Supplies Three Separate Outlets

The first step is to convert Figure 4-2 into a functional diagram as shown in Figure 4-3. Here the stages correspond to points in space, the state variable is the amount of water available as input to stage i , Q_i , and the decision variable is the amount of water to be released at outlet i , q_i . The transformation function is

$$\tilde{Q}_i = Q_i - q_i \quad i = 1, 2, 3 \quad (4.12)$$

The incidence identities are

$$Q_i = \tilde{Q}_{i+1} \quad i = 1, 2 \quad (4.13a)$$

such that

$$Q_i = Q_{i+1} - q_{i+1} \quad i = 1, 2 \quad (4.13b)$$

and the recursive equation is

$$f_i(Q_i) = \text{Max}_{q_i} \{ r_i(Q_i, q_i) + f_{i-1}(Q_{i-1}) \} \quad i = 1, 2, 3 \quad (4.14)$$

subject to

$$0 \leq \sum_{i=1}^3 q_i \leq 3 \quad (4.15)$$

where f_0 is defined as zero. Because of the boundary conditions \tilde{Q}_1 is equal to zero, and therefore $q_1 = Q_1$.

The benefit function for each outlet is given in Table 4-1 for a grid spacing of 1 ac-ft.

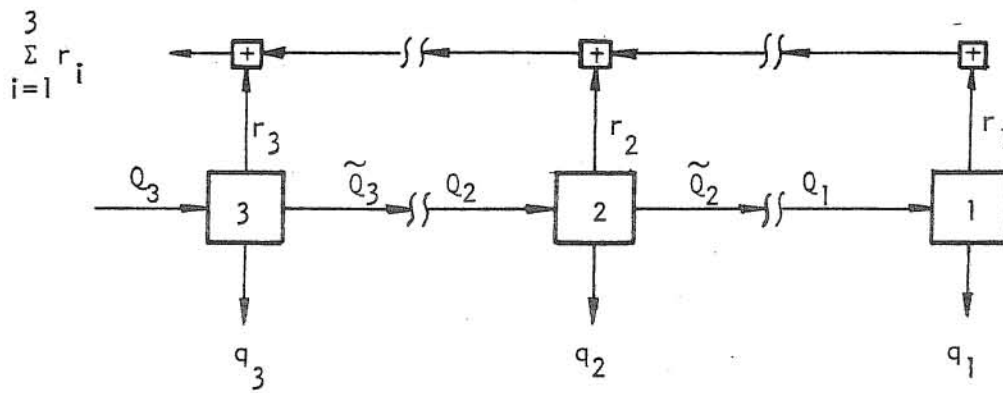


FIGURE 4-3. Functional Diagram for Pipe System in Figure 4-2.

TABLE 4-1. Benefit Functions for Pipe System

q_i ac-ft	Benefits		
	$r_1(q_1)$ \$	$r_2(q_2)$ \$	$r_3(q_3)$ \$
1	2	1	4
2	5	4	5
3	6	7	6

To apply the dynamic programming approach, start with the downstream user, designated as stage 1, and work backwards one stage at a time. For the first outlet the return obtained is

$$f_1(Q_1) = \text{Max.}_{q_1} r_1(Q_1, q_1) \quad (4.16)$$

subject to

$$0 \leq Q_1 \leq 3 \quad (4.17)$$

$$0 \leq q_1 \leq Q_1 \quad (4.18)$$

The results for various values of q_1 are given in the first two columns of Table 4-1.

Next consider stage 2. The total discharge must now be shared

between stages 1 and 2. To find the best allocation for each value of Q_2 all feasible distributions are examined in which the available flow is now divided between outlet 1 and 2. In this manner the maximum return from the two users is found for each value of Q_2 . The recursive equation now becomes

$$f_2(Q_2) = \text{Max.}_{q_2} \{r_2(Q_2, q_2) + f_1(Q_1)\} \quad (4.19)$$

subject to

$$0 \leq Q_2 \leq 3 \quad (4.20)$$

$$0 \leq q_2 \leq Q_2 \quad (4.21)$$

The value of $f_2(Q_2)$ for various values of Q_2 , Q_1 , and q_2 are shown in Table 4-2 where the $f_1(Q_1)$ values are the values from column 2 of Table 4-1.

TABLE 4-2. Computation Results for Stage 2

Q_2 ac-ft	q_2 ac-ft	$q_1=Q_1$ ac-ft	$r_2(Q_2, q_2)$ \$	$f_1(Q_1)$ \$	$f_2(Q_2)$ \$
1	1	0	1	0	1
	0	1	0	2	2
2	2	0	4	0	4
	1	1	1	2	3
	0	2	0	5	5
3	3	0	7	0	7
	2	1	4	2	6
	1	2	1	5	6
	0	3	0	6	6

Now consider stage 3. The optimal way of allocating water between stages 1 and 2 is given by equation (4.19) and the optimal return is shown by the boxed values in Table 4-2 for each value of Q_2 . Therefore the input state at stage 3 will be shared between the release q_3 and the first two stages in

an optimal fashion. The optimal return for this case is

$$f_3(Q_3) = \max_{q_3} \{r_3(Q_3, q_3) + f_2(Q_2)\} \quad (4.22)$$

subject to

$$0 \leq Q_3 \leq 3 \quad (4.23)$$

$$0 \leq q_3 \leq Q_3 \quad (4.24)$$

The value of $f_3(Q_3)$ for various values of Q_3 , Q_2 , and q_3 are shown in Table 4-3 where the $f_2(Q_2)$ values are optimal values from Table 4-2. The boxed values in Table 4-3 represent the optimal return for various values of Q_3 . The maximum of these values is shown by the shaded box and is equal to \$9. Therefore, the optimal policy is to allocate 2 ac-ft of water to outlet 1, no water to outlet 2, and 1 ac-ft to outlet 3.

TABLE 4-3. Computation Results for Stage 3

Q_3 ac-ft	q_3 ac-ft	Q_2 ac-ft	$r_3(Q_3, q_3)$ \$	$f_2(Q_2)$ \$	$f_3(Q_3)$ \$
1	1	0	4	0	4
	0	1	0	2	2
2	2	0	5	0	5
	1	1	4	2	6
	0	2	0	5	5
3	3	0	6	0	6
	2	1	5	2	7
	1	2	4	5	9
	0	3	0	7	7

If there were only 2 ac-ft of water available, the optimal policy would be to allocate 1 ac-ft of water to outlet 3, no water to outlet 2, and 1 ac-ft to outlet 1 for a return of \$6.

The use of a grid spacing of 1 ac-ft assumes that water can only be allocated in units of 1 ac-ft. If the water can be allocated in amounts less than 1 ac-ft, then the grid size could be reduced. The only difference between the analysis for a smaller grid size and the example here is that more computations would be required for the smaller grid size.

4-5. Present Value of Resource

In water resources projects the benefits occur over a long period of time, so that future benefits have to be discounted to estimate the present value of the resource. The recursive equation can easily be extended to include this feature. This is accomplished by multiplying the future return $f_{i-1}(s_{i-1})$ by the present worth factor $1/(1+\eta)$ where η is the interest rate per unit stage, and adding the result to the benefit obtained from the present stage. Therefore,

$$f_i(s_i) = \max_{d_i} \left\{ r_i(s_i, d_i) + \frac{1}{1+\eta} f_{i-1}(s_{i-1}) \right\} \quad (4.25)$$

In general, as the number of stages increases, that is, as stage 1 extends further into the future, its return will have a decreasing influence on the present value of the resource, and hence on the optimal operating policy. The recursive equation therefore has the property of converging on a steady state policy as the number of stages is increased. The steady state condition may be explained by considering a system with a number of stages which extend to infinity. For this case the recursive equation becomes

$$f(s) = \lim_{N \rightarrow \infty} f_N(s_N) = \lim_{N \rightarrow \infty} \max_{d_N} \left\{ r_N(s_N, d_N) + \frac{1}{1+\eta} f_{N-1}(s_{N-1}) \right\} \quad (4.26a)$$

or

$$f(s) = \max_d \left\{ r(s, d) + \frac{1}{1+\eta} f(s) \right\} \quad (4.26b)$$

That is, it will be found that the optimal returns $f_N(s_N) \rightarrow f(s)$ and the optimal decisions $d_N \rightarrow d$ become independent of the stage number if there are a large number of stages remaining in the planning period. The optimal stage invariant solution, $f(s)$ and d in this case, is termed the steady state solution.

4-6. Probabilistic and Stochastic Models

The shortcomings of the deterministic model become increasingly evident when one attempts to obtain an optimal design and operating rule for a water resources system. In actual practice the inflows are not completely known. Hence the simple deterministic model should be replaced by a more sophisticated one in which the inflows are described by a set of probability distributions.

The recursive equation for estimating the maximum expected benefits assuming the inflow varies according to a known probability distribution may be written as

$$f_i(s_i) = \max_{d_i} \left\{ r_i(s_i, d_i) + \frac{1}{1+\eta} \sum_{k=1}^M P_k f_{i-1}(s_{i+1}) \right\} \quad (4.27)$$

where the P_k 's are the probabilities for discrete values of the inflow q_{ik} during the period i in the transformation function

$$\tilde{s}_i = s_i + q_{ik} - d_i - e_i \quad (4.28)$$

Equation (4.27) provides an estimate of the maximum expected benefit, but no indication is given regarding its variability or the risks involved by the particular policy. Further, the equation is only suitable when a zero, or negligible, serial correlation exists between flows. In general this is not the case, so the inflows should not be characterized by a simple probability distribution. In these circumstances, the equation should be rewritten in

terms of conditional probabilities.

The Monte Carlo procedure is favored by some engineers for the generation of stream flow because of the greater risk information which it provides in the analysis. That is, by considering a set of equally likely streamflow sequences the dispersion of the maximum benefits may be obtained for each level of development. The optimum scale of development may therefore be selected by considering both the expected benefits and the degree of risk that the project will fail to break even [Hall and Howell, 1963].

Butcher [1968] states that if a first order Markov chain is used in the analysis it is possible to derive the optimal release policy for a system. The number of stages, however, should be sufficient to remove the influence of the starting conditions. Expressing the conditional probabilities in the form $P(q_{i-1}/q_i)$, which is the probability of q_{i-1} occurring given q_i , the recursive equation now becomes

$$q_i = q_i \max$$
$$f_i(s_i, q_i) = \max_{d_i} \left\{ r_i(s_i, d_i) + \frac{1}{1+\eta} \sum_{q_{i-1}=0} P(q_{i-1}/q_i) f_{i-1}(s_{i-1}, q_{i-1}) \right\} \quad (4.29)$$

where there are two state variables, s_i and q_i .

Pilot calculations indicate that this method of solution is fairly compact, and that the releases tend to approach a steady state condition in a small number of iterations.

The use of stochastic models in hydrologic analysis has increased rapidly in recent years, and it seems likely that stochastic techniques will form an integral part of the more rational mathematic models now being developed.

4-7. Nonserial Systems

As explained previously, dynamic programming is normally considered

suitable for processes having serial structures. However, the technique may also be extended to include the analysis of branched systems.

A nonserial system is indicated in the functional diagram in Figure 4-4. Recently, methods have been developed for decomposing a nonserial system into an equivalent serial system which is solvable by dynamic programming [Meier and Beightler, 1967]. In such a system one of the stages (K) of a subsystem has in addition to its serial input, another input which is the final output from a different sequence of M serial stages forming a converging branch. The problem is to maximize the sum of the returns from all M + N stages. This is accomplished by treating the output from the cut state as a parameter.

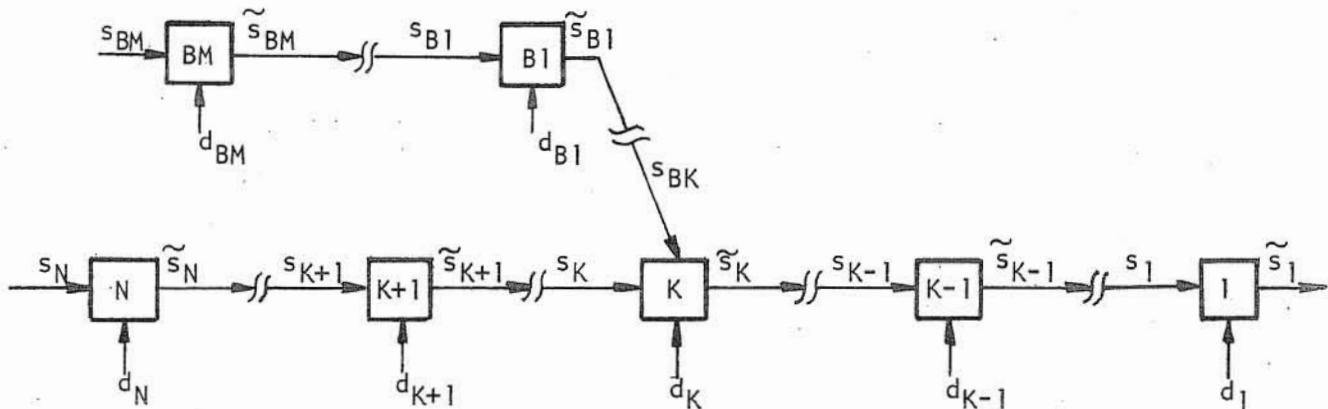


FIGURE 4-4. Functional Diagram for Converging Nonserial System

If the output of the branch system is maximized the branch flow entering the main system is fixed through the transformation function. The value of the flow which optimizes the branch system alone will not generally be the optimal flow for the main stem system. However, if the optimal branch output is found as a function of the branch flow entering the main stem, it

can be combined with the output of the main stem system. The optimal value of the branch flow can then be found by considering its effect on both the branch and main stem stages.

4-8. Applications of Dynamic Programming to Water Resources Systems

The dynamic programming approach has been used to study several types of water resources systems. Buras [1966] provides a detailed account of the methodology and discusses applications to water storage, conjunctive use of ground and surface water supplies, hydroelectric power production, and watershed management.

Hall and Buras [1961] were the first to propose that dynamic programming be applied to the optimization of reservoir systems. They studied the problem of allocating stored water to various purposes. Hall [1964], Young [1967], and Hall, Butcher, and Esagbue [1968] have reported studies concerned with optimizing individual multipurpose reservoirs. In these studies, the stages are defined as time periods and the state variables describe the portions of the reservoir resources allocated to satisfy conflicting demands. Hall was concerned with finding the optimum reservoir size. Young and Hall, et al., were concerned with determining the optimum operation rules for a reservoir. Other investigators [Amir, 1967; Buras, 1965; Meier and Beightler, 1967; Schweig and Cole, 1968] have been concerned with multi-reservoir systems wherein the reservoirs play the role of stages and the state variables involve flow or discharge quantities.

Two types of algorithms may be used to investigate optimization problems. The algorithms may be designed to move forward in time from the start of the planning period, or to work recursively backwards in time from the end of the planning period. Hall, for example, has shown a preference for the backward algorithm. Others, such as Young [1967], have developed models which

move forward in the direction of time.

With regard to stochastic models, Hall appears to favor the Monte Carlo method for streamflow generation. Using this technique it is possible to check the system response when subjected to a number of equally likely streamflow sequences. This gives a measure of the output variability, and hence the degree of risk. Butcher [1968], on the other hand, suggests that a first order Markov chain be incorporated in the solution algorithm if serial correlation is present in the streamflows. In this case the streamflow in succeeding time periods is connected to current flow by a matrix of conditional probabilities.

The conjunctive use of surface and ground water reservoirs is now a well established practice in water supply. This problem has been studied by Buras and Hall [1961] and Buras [1963]. These studies were made to determine the optimum capacity of the surface reservoir or to determine the best operating policy for the system. Buras and Bear [1964] used dynamic programming to establish the optimal pumping capacity and mode of operation for a coastal aquifer.

In the field of irrigation, Hall [1961] used the dynamic programming approach to determine the optimal allocation of water to a number of homogeneous areas located in sequence along a main supply line. Using a somewhat different basis, Flinn and Musgrave [1967] designed a model which was capable of specifying the optimal allocation of irrigation water during the growing season. With such a model an irrigation authority could estimate the optimal release policy in terms of intra-seasonal variations in water cost and crop price. Hall and Butcher [1968] have also proposed a dynamic programming approach to determine intra-season allocation of irrigation water.

The general problem of hydro-power output optimization has been

studied by a number of analysts. As an example, Hall and Roefs [1966] studied the problem of optimal releases for power generation given complete knowledge of the inflow hydrology.

One of the more recent studies by Hall and Shephard [1967] involved the analysis of a complex water resources system for multiple use of water. The system, including 10 reservoirs and 4 major streams, is located on the Sacramento River in Northern California. The problem was to maximize an objective function which depended on the sale of firm and off-peak energy and seasonal and off-seasonal water. This was accomplished by combining dynamic and linear programming algorithms. Although a deterministic hydrology was assumed in this study, some suggestions were made for analyzing the system with a stochastic input.

Liebman and Lynn [1966] used dynamic programming to assign waste treatment requirements to a group of dischargers so as to minimize the total overall cost of waste treatment along a stream. Stages were defined as sequential reaches of a river, and state variables were employed to define water quality as measured by dissolved oxygen. Dysart and Hines [1969] used dynamic programming to determine the minimum cost abatement policy to obtain specific stream standards. The stages were defined as sequential reaches of the river. The decision variables at each stage were the level of cooling of heated waste and the level of treatment of organic waste. The state variables were water temperature, dissolved oxygen, and biological oxygen demand. The Chattahoochee River basin was used as a numerical illustration.

Erickson et al., [1968] optimized step aeration waste treatment systems. The objective was to determine the allocation of influent and the distribution of volume among a series of reactors which minimized the total aeration volume required to achieve a specified level of treatment.

Joeres [1967] used a deterministic dynamic programming model to find the minimum total cost of control measures and damage costs for sedimentation in a hypothetical river basin.

5. OTHER TECHNIQUES

Network theory, game theory, and simulation can be used in water resources system analysis. Each technique relies heavily on mathematical modeling to assess the technical and economic optimality of alternative system designs, policies, and operating procedures.

5-1. Network Theory

Network analysis has long played an important role in electrical engineering; however, in recent years there has been a greater use of certain concepts and tools of network theory in other contexts including water resources systems analysis. Network theory can be useful in the solution of problems which involve allocating flows in order to maximize the flow through a network connecting a source and a destination, finding the shortest route through a network, or choosing a set of connections that provide a route between any two points in a network in such a way as to minimize the total length of these connections.

In order to introduce some terminology and definitions used in network theory, we must first briefly discuss some elementary topics in the theory of graphs.

According to the terminology of the theory of graphs [Ford and Fulkerson, 1962], a graph is a set of junction points called nodes with certain pairs of these nodes being joined by lines called branches (or edges, arcs, or links).

A network is a graph with a flow of some type in its branches. If there is a limit to the magnitude of the flow in any branch of a network, then a capacity restriction is imposed on that branch. A chain between nodes i and j is a sequence of branches connecting these two nodes. When the direction of

travel along the chain is specified, the chain is called a path. An arc of a network with a positive capacity (flow) in one direction only is said to be directed (or oriented), and a directed network is one in which all arcs are directed. A path which connects a node with itself is called a loop (or cycle). A connected graph has a chain connecting every pair of nodes. A tree is a connected graph which has no loops.

A node is called a source if each one of its branches is directed such that the flow moves away from that node. Similarly, a node is a sink if each of its branches is directed such that the flow moves toward that node.

Consider a network consisting of a single source, a single sink, and some intermediate nodes. Assume that the network has capacity restrictions $d_{ij} \geq 0$ on each arc (i, j) . The flow in the network can be considered as fluid, money, electricity, traffic, time, etc. We wish to determine a procedure for computing the maximum feasible steady-state pattern of flows from the source to the sink. This is the form, then, of an optimization problem. If the sum of all flows through an arc equals its capacity, then, the arc is saturated. The difference between its capacity and the flows through it is called its residual flow. To find the maximal flow in the network, the technique of maximal flow-minimal cut is often used. A collection of arcs such that each chain from the source to the sink contains at least one arc of the collection is called a cut; clearly no network flow can be larger than the sum of all capacities of the arcs of a cut. It follows that if a network flow can be found which equals the total of the capacities of the arcs of a cut, then that network flow is maximal, and the cut capacity is minimal. Algorithms for the solution of this problem are available [Ford and Fulkerson, 1962; Hillier and Lieberman, 1967].

Many network-flow problems can be reformulated as linear programming or dynamic programming problems [Ford and Fulkerson, 1962; Fulkerson, 1961, 1963].

5-2. Game Theory

Game theory is a mathematical theory that deals with the general features of competitive situations in a formal, abstract way. Game theory is sometimes called statistical decision theory.

Games of strategy deal with situations where there are conflicts of interest between two or more 'persons' (persons may be people, companies, countries, etc). Generally, games involve conflict situations in economic, social, political, or military activities. Games of strategy assume that players can influence the final outcome, and hence that the outcome is not controlled purely by chance. A game is a set of rules for playing. These rules describe the moves, who makes moves, when they are made, what information is available to each of the players, what terminates a play of the game, etc. After a play is terminated, a pay-off is achieved. If the sum of the pay-offs to all participants at the end of the play is zero, then the game is called a zero-sum game. Two-person games involve a conflict of interest between two persons. A zero-sum, two-person game is one in which, at the end of a play, one person gains what the other person loses [Williams, 1954].

The concept of a strategy is very important in game theory. If we imagine that one player writes out what he will do under all possible circumstances (moves of his opponent) at each move in the play of a game, then we can visualize the meaning of a strategy. In two-person games, it is permissible for nature to make some of the moves, so long as the players can influence the outcome. When some of the moves are determined by chance, the outcome of the game is not strictly determined, and we can only talk about the expected outcome. If player 1 has m strategies and player 2 has n strategies, the pay-offs, a_{ij} , can be arranged into an $m \times n$ matrix.

If player 1 chooses strategy i , he is sure of getting $\min_j \{a_{ij}\}$, no matter what player 2 does. The optimal decision, then, for player 1 is,

$\max_i \{ \min_j \{ a_{ij} \} \}$. Player 2 is attempting to prevent player 1 from getting any more than is necessary. Therefore his best strategy is determined by,

$\min_j \{ \max_i \{ a_{ij} \} \}$. If it turns out that there is an element a_{rk} such that,

$$a_{rk} = \max_i \{ \min_j \{ a_{ij} \} \} = \min_j \{ \max_i \{ a_{ij} \} \} \quad (5.1)$$

the game is said to have a saddle point. The optimal strategy for player 1 is r and for player 2 the optimal strategy is k . The difficulty arises when no saddle point can be found in the matrix. In this case, a 'mixed strategy' is called for. In the case of mixed strategies, we can only speak of expected pay-offs. The objective of player 1 becomes to maximize the 'expected' pay-off and of player 2 to minimize the 'expected' pay-off.

Many zero sum, two person games can be solved by a linear programming approach [Hadley, 1962].

Game theory has also been extended to include n -person games and non zero sum games.

Maass, et al. [1962] present an example of how game theory can be used in water resources systems analysis. The example used is a reservoir to be used for flood protection and to supply irrigation water. The decision maker is faced with conflicting objectives in that for maximum flood protection he would want to spill a large quantity so as to increase his capacity for collecting flood runoff, but he would want to retain a large quantity for irrigation use during the next season. Game theory is used to show how the decisions would be made using expected outcomes.

5-3. Simulation

Simulation models are used when the interrelationships between relevant parameters or system constraints are too complex to be solved

analytically. Simulation models consist basically of functional relationships which describe system characteristics as indicated by the response of selected output variables to the input variables. The physical models which have traditionally been employed by hydraulic engineers to study flows and currents in engineered and natural systems are simulation models.

In water resources systems analysis, simulation consists of formulating functional relationships and then performing calculations to explore numerically the response of output variables to input variables. Simulation does not directly yield information such as the optimal capacity of a reservoir; rather the output from a simulation model is used to construct a response surface which can be examined to determine an optimum solution as indicated by maximum and minimum points of the quantities of interest. Simulation models generally require fewer simplifying assumptions than analytical models such as linear or dynamic programming models. An approach often employed is to use an approximate analytical model to define the region of near optimality and then to converge on the optimal solution through simulation.

A lengthy system description, a computer program, and computer output data are necessary to illustrate a simulation model.

Lewis and Shoemaker [1962] report a study of hydroelectric power by the North Pacific Division Office of the Corps of Engineers. The program can handle up to seventy hydroelectric projects. It computes power generation at each project in the system, tests for limits, adjusts regulation where necessary to avoid violation of these limits, regulates storages to produce energy equal to the specified system load, and derives additional information that is used in system regulation analysis.

Hufschmidt and Fiering [1966] have presented guidelines for constructing simulation models of water resources systems. These guidelines are

illustrated by a simulation of the Lehigh River Basin. Hydroelectric power, water supply, recreation, and flood control are the purposes considered in the simulation.

Meredith [1968] used simulation techniques to study the economic benefits for a multipurpose reservoir for water supply, recreation, and flood control. Simulation has also been used to study the impact of water on the growth and development of a region [Hamilton, 1968].

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